Journal of Statistical Physics, Vol. 25, No. 4, 1981

# Multiple Scattering in Random Media II. The General 2BA for the Uncorrelated System

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*Received August 5, 1980*

The paper is an application of a general microscopic approach to the theory of the average scattering matrix for a particle interacting with random scatterers. We present a detailed treatment for the case of uncorrelated positions of the scatterers. First, the general two-body additive approximation is used to truncate the hierarchy of correlation functions for fluctuations. It is shown that the self-energy is accurate through the fourth power of the individual scattering amplitude. Second, the hierarchy is terminated at the next stage. The self-energy is correct to the sixth power of the scattering amplitude.

**KEY WORDS:** Multiple scattering; random media.

## 1. INTRODUCTION

We summarize the basic equations of the microscopic approach to multiple scattering. In the notation of I,[1] with 2 standing for the wave vector $k_2$, and with a matrix notation in wave vector space, the self-energy corresponding to the ensemble averaged $T$ matrix is given by

$$\Sigma(2) = N\langle 2|(1-\overline{K})^{-1}t|2\rangle \tag{1}$$

The kernel is $\overline{K} = \overline{K}_0 + \overline{K}_1$. $\overline{K}_0$ is the quasicrystalline approximation part

$$\langle 1|\overline{K}_0(2)|3\rangle = \langle 1|tG_0|3\rangle \overline{F}_2(2-3) \tag{2}$$

where $\overline{F}_2(2-3)$ is the Fourier transform of the static pair distributions, and

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$\bar{K}_1$ is a kernel that takes account of the fluctuations. It is given by

$$\langle 1|\bar{K}_1(2)|3\rangle = \frac{\langle 1|tG_0|2-\underline{\lambda}\rangle}{N}\Delta(\underline{\lambda}\,|\,0)\sum_\beta \delta\,\overline{E'_\beta(\lambda)\,\langle 2-\underline{\lambda}|\Gamma_\beta|3\rangle} \tag{3}$$

Here

$$E^1_\beta(\lambda) = \sum_{\gamma\neq\beta} E_{\gamma\beta}(\lambda) = \sum_{\gamma\neq\beta}\exp\left[i\lambda(R_\beta - R_\gamma)\right]$$

and the $R_\beta$ are the site positions. This is an ensemble average involving a microscopic fluctuation $\Gamma_\alpha$. The manipulation of the fluctuation equation to exhibit collective effects gave

$$\langle 1|\Gamma_\alpha(2)|3\rangle = \langle 1|\delta J_\alpha(2)|3\rangle + \delta\sum_{\beta\neq\alpha}\langle 1|\delta L_{\alpha\beta}\,\Gamma_\beta|3\rangle \tag{4}$$

The source term $\delta J_\alpha$ and the matrix $\delta L_{\alpha\beta}$ are given in terms of

$$\langle 1|(K_0)_{\alpha\beta}|3\rangle = N\langle 1|tG_0|3\rangle E_{\alpha\beta}(2-3)(1-\delta_{\alpha,\beta}) \tag{5}$$

The source term $\delta J_\alpha$ is

$$\delta J_\alpha = \frac{\delta K^0_\alpha}{N} + \frac{1}{N^2}\,\frac{\bar{K}_0}{1-\bar{K}_0}\sum_\gamma \delta K^0_\gamma \tag{6}$$

The first term is a direct term of order $t$ with the functional form of the restricted 2BA. The second term is of order $t^2$ and is a collective term involving pairs of particles other than $\alpha$.

The kernel $\delta L_{\alpha\beta}$ is

$$\delta L_{\alpha\beta} = \frac{1}{N}(\delta K_0)_{\alpha\beta} + \frac{1}{N^2}\,\frac{\bar{K}_0}{1-\bar{K}_0}\,\delta K^1_\beta \tag{7}$$

Again the direct term is of order $t$ and the collective term is of order $t^2$. In the restricted 2BA we limited ourselves to an ansatz for $\Gamma_\alpha$ of the same form, replacing the direct source term $\langle 1|tG_0|3\rangle$ by a quantity $\langle 1|H|3\rangle$. $H$ was determined by using a hierarchy equation to treat nonlinear fluctuations. For the uncorrelated case and for a one-dimensional $\delta$ function we found

$$\langle 1|H|3\rangle = tG^*_0(3), \qquad G^*_0(3) = G_0(3)/\left[1 - NtG_0(3)\right]$$

We now want to improve on this result. To this end we first provide motivation for the general 2BA. Thus an expression for $\Gamma_\alpha$ accurate to order $t^2$ is

$$\Gamma_\alpha = \delta J_\alpha + \frac{1}{N^2}\delta\sum_{\beta\neq\alpha}(\delta K_0)_{\alpha\beta}\sum_{\gamma\neq\beta}(\delta K_0)_{\beta\gamma}$$

Let us examine the functional form of the second term. It has three distinct types of contributions. The first type comes from particle reduction $\gamma = \alpha$.

It yields a generalized 2BA functional form. For example for the uncorrelated system it is

$$\langle 1|tG_0|3-\underline{\lambda}\rangle\langle 3-\underline{\lambda}|tG_0|3\rangle \sum_{\beta\neq\alpha} E_{\alpha\beta}(\lambda)$$

The second type of contribution involves $\alpha\neq\beta\neq\gamma$ with a wave vector reduction. For the uncorrelated case it is $N\langle 1|tG_0|3\rangle\langle 3|tG_0|3\rangle\sum_{\gamma\neq\alpha}E_{\alpha\gamma}\times(2-3)$. This is of the restricted 2BA form, and is in fact the first term in the expansion of the form for $\langle 1|H|3\rangle$. The third type of functional dependence has an irreducible three-body additive form.

The functional form of $\Gamma_\alpha$ is used to compute the kernel $\overline{K}_1$. In the uncorrelated system it is found that the 3BA terms give zero contribution. In fact the collective part of the source term also gives no contribution. Thus the functional form of the general 2BA is adequate to give $\overline{K}$ to $t^3$ accuracy and with it a self-energy accurate to order $t^4$. The 3BA terms give a finite contribution for the case of general correlations. The general 2BA together with the collective source term give only part of the $t^4$ contributions. In the present paper we work out the details of the general 2BA for the uncorrelated case.

## 2. THE RESTRICTED 2BA SELF-ENERGY

We first present some further details of the restricted 2BA for the uncorrelated case. The quantities encountered play a role in the general 2BA. We pay particular attention to the one-dimensional $\delta$-function case where $\langle 1|t|\underline{2}\rangle = t = \tau/L$. This case was extensively treated by Klauder.[2] The kernel $\overline{K}_1$ is then

$$\overline{K}_1(2;3) = \frac{n\tau}{L}^2 G_0^*(3)G_0(3) \tag{8}$$

where $n = N/L$ is the linear density.

The self-energy is

$$\Sigma = n\tau/\left[1-\overline{K}_1(2;3)\right] \tag{9}$$

where

$$\tau = v_0 \bigg/ \left(1 - \frac{v_0}{2\pi}\int \frac{d\lambda}{E+i\epsilon-\lambda^2/2}\right) \tag{10}$$

We first write $\overline{K}_1$ in a form to make a direct correspondence with Klauder's results. We have

$$\overline{K}_1(2;\underline{3}) = \frac{n\tau^2}{2\pi}\int d\lambda\, G_0^*(\lambda)G_0(\lambda) = \frac{\tau}{2\pi}\int d\lambda\{G_0^*(\lambda) - G_0(\lambda)\} \tag{11}$$

$$\Sigma = n\left[\frac{1}{v_0} - \frac{1}{2\pi}\int d\lambda\, G_0^*(\lambda)\right]^{-1} \tag{12}$$

This is a form intermediate between Klauder's fourth and fifth approximations.

In the present theory the results are expressed most simply in terms of the $t$ matrix rather than the bare potential. We encounter the spatial form of the Green's function. Let

$$\tilde{G}_0(x) = \int e^{i\lambda x} G_0(\lambda)\, d\lambda$$
$$G_0(\lambda) = \frac{1}{2\pi} \int e^{-i\lambda x} \tilde{G}_0(x)\, dx \tag{13}$$

The explicit forms are

$$G_0(\lambda \mid E) = \frac{1}{E + i\epsilon - \lambda^2/2}$$
$$\begin{aligned} \tilde{G}_0(x \mid E) &= -\left[2\pi/(2|E|)^{1/2}\right]\exp\left[-|x|(2|E|)^{1/2}\right], \qquad E < 0 \\ &= -\left[2\pi i/(2E)^{1/2}\right]\exp\left[i|x|(2E)^{1/2}\right], \qquad E > 0 \end{aligned} \tag{14}$$

The starred Green's functions are more complicated. We have

$$\begin{aligned} G_0^*(\lambda \mid E) &= G_0(\lambda \mid E - n\tau(E)) \\ \tilde{G}_0^*(x \mid E) &= \tilde{G}_0(x \mid E - n\tau(E)) \end{aligned} \tag{15}$$

We need to examine the quantity $E - n\tau(E)$ in more detail. For $E > 0$

$$\tau(E) = v_0\left[1 - \frac{iv_0}{(2E)^{1/2}}\right]\frac{1}{(1 + v_0^2/2E)} \tag{16}$$

so that

$$E - n\tau(E) = E - \frac{nv_0}{1 + v_0^2/2E} + \frac{iv_0^2 n}{(2E)^{1/2}(1 + v_0^2/2E)} \tag{17}$$

This has a finite positive imaginary part for both signs of $v_0$. Thus we have

$$\tilde{G}_0(x \mid E - n\tau) = -\frac{2\pi i}{\left[2(E - n\tau)\right]^{1/2}}\exp\left[i|x|2^{1/2}(E - n\tau)^{1/2}\right] \tag{18}$$

exhibiting a damped oscillation in space.

For $E < 0$ there are two cases. We work with $E + i\epsilon$ with $\epsilon$ small and positive. Then $\tau(E)$ is nearly real. To first order in $\epsilon$

$$E - n\tau(E) = -|E| - \frac{nv_0}{1 + v_0/(2|E|)^{1/2}} + i\epsilon + \frac{i\epsilon n v_0^2|E|^{-3/2}}{(1 + v_0/2^{1/2}|E|^{1/2})^2} \tag{19}$$

The imaginary part is always positive while the real part can be positive or negative. As $\epsilon \to 0$ we have ($E < 0$, $E - n\tau < 0$),

$$\tilde{G}_0(x \mid E - n\tau) = -\frac{2\pi}{2^{1/2}|E - n\tau|^{1/2}} \exp\left[-|x|(2|E - n\tau|)^{1/2}\right] \tag{20}$$

With the spatial form of the Green's functions the restricted 2BA gives a formula

$$\bar{K}_1(2;\underline{3}) = \frac{n\tau^2}{(2\pi)^2} \int \tilde{G}_0(x \mid E - n\tau)\tilde{G}_0(x \mid E)\, dx \tag{21}$$

We have in the three regions

$$\begin{aligned}
&E > 0, && \bar{K}_1(2;\underline{3}) = \frac{(-i)n\tau^2}{(2E)^{1/2}(E - n\tau)^{1/2}} \\
& && \times \left[\frac{1}{E^{1/2} + (E - n\tau)^{1/2}}\right] \\
&E < 0, \quad E - n\tau < 0 && = \frac{n\tau^2}{(2|E|)^{1/2}} \frac{1}{|E - n\tau|^{1/2}} \\
& && \times \left[\frac{1}{|E|^{1/2} + |E - n\tau|^{1/2}}\right] \\
&E < 0, \quad E - n\tau > 0 && = \frac{(i)n\tau}{(2|E|)^{1/2}(E - n\tau)^{1/2}} \\
& && \times \left[\frac{1}{|E|^{1/2} - i(E - n\tau)^{1/2}}\right]
\end{aligned} \tag{22}$$

Finally, we examine $E - n\tau(E)$, contrasting the behaviors for repulsive and attractive $v_0$. For $E > 0$ we use Eq. (17) for $E - n\tau(E)$.

If $v_0 < 0$ (attractive potential) the value at $E = 0$ is zero and the slope of the real part is positive for all $E$. So the real part $E - n\tau(E)$ is a monotonic function tending to $E$ as $E \to \infty$. If $v_0 > 0$ (repulsive potential) there are two cases at high densities when $2n > v_0$ the slope at $E = 0$ is negative and $\mathrm{Re}[E - n\tau(E)]$ has a negative region, rising to zero at $E = \frac{1}{2}v_0(2n - v_0)$. In the low-density case $2n < v_0$ the function is positive and monotonic.

If $E < 0$ the value of $E - n\tau(E)$ is always negative for the repulsive case, viz.

$$E - n\tau = -\left[|E| + \frac{nv_0(2|E|)^{1/2}}{(2|E|)^{1/2} - v_0}\right]$$

For $E < 0$ and $v_0 < 0$ (attractive potential) we have

$$E - n\tau = -\left[ |E| - \frac{n|v_0|(2|E|)^{1/2}}{(2|E|)^{1/2} - |v_0|} \right] \tag{23}$$

Thus $E - n\tau$ starts out at zero for $E = 0$, has a negative slope and tends to $-\infty$ as one approaches the bound state $|E| = v_0^2/2$. If $|E|$ is slightly larger than this, $E - n\tau$ tends to $+\infty$. It is then monotonic decreasing, passing through zero at

$$(2|E|)^{1/2} = \tfrac{1}{2}|v_0| + \left(\tfrac{1}{2}v_0^2 + 4n|v_0|\right)^{1/2} \tag{24}$$

and tending to $-|E|$ as $|E| \to \infty$.

## 3. GENERAL 2BA—UNCORRELATED DELTA-FUNCTION POTENTIALS

We first adapt Eq. (61)–(64) of I for $\langle 1|\overline{U}_2(\lambda)|3\rangle$ to the uncorrelated case. Recall that

$$\overline{U}_2(\lambda) = \sum_\beta \overline{E_\beta^1(\lambda)\Gamma_\beta}$$

$$\overline{U}_3(\lambda \| \lambda_1) = \sum_{\gamma \neq \beta \neq \alpha} \overline{E_\gamma(-\lambda - \lambda_1)E_\beta(\lambda)E_\alpha(\lambda_1)\Gamma_\alpha} \tag{25}$$

with the kernel $\overline{K}_1$ given as

$$\langle 1|\overline{K}_1(2)|3\rangle = (1/N)\langle 1|tG_0|2 - \underline{\lambda}\rangle\Delta(\underline{\lambda}\,|\,0)\langle 2 - \underline{\lambda}\,|\,\overline{U}_2(\lambda)|3\rangle \tag{26}$$

We write the equation for $\overline{U}_2(2-3)$ separately. In the $N \to \infty$ limit, the integral term, the collective part of the source term, and $\overline{C}_2(2-3)$ all vanish.

$$\begin{aligned}\langle 1|\overline{U}_2(2-3)|3\rangle &- N\langle 1|tG_0|3\rangle\langle 3|\overline{U}_2(2-3)|3\rangle \\ &= N^2\langle 1|tG_0|3\rangle + \langle 1|tG_0|2 - \underline{\lambda}_1\rangle \\ &\quad \times \langle 2 - \underline{\lambda}_1|\overline{U}_3(3-2\| \lambda_1)\rangle\Delta(\underline{\lambda}_1|\lambda)\Delta(\underline{\lambda}_1|2-3)\end{aligned} \tag{27}$$

One finds that the $\overline{U}_3$ term is zero for the restricted 2BA with or without collective terms. The general 2BA uses the microscopic truncation

$$\langle 1|\Gamma_\alpha(2)|3\rangle = \langle 1|H(3)|3\rangle E_\alpha^0(2-3) + \langle 1|H(2-\underline{\lambda})|3\rangle E_\alpha^0(\underline{\lambda})\Delta(\underline{\lambda}\,|\,2-3) \tag{28}$$

If one sets up the hierarchy equations for $\overline{U}_3$, one sees that the source term vanishes for the uncorrelated system. So the exact $\overline{U}_3$ is at most of order $t^2$, contributing only terms of order $t^3$ to $\overline{U}_2$.

In the equation for $\overline{U}_2(\lambda)$, $\lambda \neq 2-3$ we treat $\overline{U}_2(2-3)$ as an inhomogeneous term. In the $N \to \infty$ limit

$$\begin{aligned}
&\langle 1|\overline{U}_2(\lambda)|3\rangle - N\langle 1|tG_0|2-\lambda\rangle\langle 2-\lambda|\overline{U}_2(\lambda)|3\rangle \\
&\qquad - \langle 1|tG_0|2-\lambda-\underline{\lambda}_1\rangle\langle 2-\lambda-\underline{\lambda}_1|\overline{U}_2(\underline{\lambda}_1)|3\rangle\Delta(\underline{\lambda}_1|2-3)\Delta(\underline{\lambda}_1|0) \\
&\quad = \langle 1|tG_0|3-\lambda\rangle\langle 3-\lambda|\overline{U}_2(2-3)|3\rangle \\
&\qquad + \langle 1|tG_0|2-\underline{\lambda}_1\rangle\langle 2-\underline{\lambda}_1|\overline{U}_3(-\lambda\|\lambda_1)|3\rangle\Delta(\underline{\lambda}_1|\lambda)
\end{aligned} \tag{29}$$

Note that $\overline{U}_2(2-3) \sim N$ and $\overline{U}_2(\lambda) \sim 1$ for $\lambda \neq 2-3$.

We now specialize to the case of the one-dimensional delta function. Then $\langle 1|\overline{U}_2(\lambda)|3\rangle$ is independent of the wave vector $k_1$. It depends parametrically on the wave vectors $k_2$ and $k_3$. We introduce

$$\langle 1|\overline{U}_2(\lambda)|3\rangle = Z(\lambda; 2; 3) \tag{30}$$

and write $Z(\lambda)$ for brevity. Passing to the continuum limit, with $t = \tau/L$, $n = N/L$, we find for $\lambda \neq 2-3$

$$\begin{aligned}
&\{1 - n\tau G_0(2-\lambda)\}Z(\lambda) - \frac{\tau}{2\pi}\int_{-\infty}^{+\infty} G_0(2-\lambda-\lambda_1)Z(\lambda_1)\,d\lambda \\
&\quad = n\tau G_0(3-\lambda)\frac{Z(2-3)}{N}
\end{aligned} \tag{31}$$

In the $N \to \infty$ limit we have

$$Z(2-3)/N = n\tau G_0^*(3) \tag{32}$$

This is unchanged from the restricted 2BA.

It is convenient to define $Y(\lambda)$ by

$$Z(\lambda) = Y(\lambda)n\tau Z(2-3)/N \tag{33}$$

Then

$$\left[1 - n\tau G_0(2-\lambda)\right]Y(\lambda) - \frac{\tau}{2\pi}\int G_0(2-\lambda-\lambda_1)Y(\lambda_1)\,d\lambda_1 = G_0(3-\lambda) \tag{34}$$

The kernel $\overline{K}_1$ is given as

$$\overline{K}_1(2;3) = \frac{n\tau^2}{L}G_0^*(3)\left[G_0(3) + \frac{\tau}{2\pi}\int G_0(2-\lambda)Y(\lambda)\,d\lambda\right] \tag{35}$$

It consists of the restricted 2BA plus a contribution from $\lambda \neq 0$.

Our task is now to study the integral equation for $Y(\lambda)$. In the units that we have used $\hbar = m = 1$. Energies are inverse lengths squared. $v_0$ and with it the complex $\tau(E)$ are inverse lengths. The quantities $n/|\tau|$ or $n/|v_0|$ are dimensionless. There are two obvious cases that correspond to $n/v_0$

large and small. For the high-density limit the first approximation to $Y(\lambda)$, holding as $n\tau$ is finite with $\tau \to 0$, is

$$Y(\lambda) = \frac{G_0(3-\lambda)}{1 - n\tau G_0(2-\lambda)} \tag{36}$$

We write $\Delta\overline{K}_1(2;3)$ for the correction to the restricted 2BA. Then

$$\Delta\overline{K}_1(2;3) \to \frac{n\tau^2}{2\pi}\left(\frac{\tau}{2\pi}\right)\int\int G_0^*(3)G_0^*(2-\lambda)G_0(3-\lambda)\,d\lambda\,d(3) \tag{37}$$

In terms of the spatial Green's functions

$$\Delta\overline{K}_1(2;3) \to \frac{n\tau^3}{2\pi}\int dx\,\tilde{G}_0^2(x\,|\,E - n\tau)\tilde{G}_0(x\,|\,E)e^{-ik_2x} \tag{38}$$

which, in contrast to the restricted 2BA, is dependent on $k_2$.

The second simple limit is a low-density limit where $n\tau \to 0$ with any value of $\tau$. We write

$$\begin{aligned} Y(\lambda) &= \frac{\tau}{2\pi}\int G_0(2-\lambda-\lambda_1)Y(\lambda_1)\,d\lambda_1 \\ &\quad - \frac{\tau}{2\pi}n\tau G_0^*(2-\lambda)\int G_0(2-\lambda-\lambda_1)Y(\lambda_1)\,d\lambda_1 \\ &= \frac{G_0(3-\lambda)}{1-n\tau G_0(2-\lambda)} \end{aligned} \tag{39}$$

and a related (adjoint) form for the quantity

$$X(\lambda) = Y(\lambda)\left[1 - n\tau G_0(2-\lambda)\right] \tag{40}$$

$$\begin{aligned} X(\lambda) &- \frac{\tau}{2\pi}\int G_0(2-\lambda-\lambda_1)X(\lambda_1)\,d\lambda_1 \\ &- \frac{\tau}{2\pi}n\tau\int G_0(2-\lambda-\lambda_1)G_0^*(2-\lambda_1)X(\lambda_1)\,d\lambda_1 = G_0(3-\lambda) \end{aligned} \tag{41}$$

In both cases we have a convolution structure when the term proportional to $nt$ is neglected. We then solve the equation

$$W(\lambda\,|\,\lambda^1) - \frac{\tau}{2\pi}\int G_0(2-\lambda-\lambda_1)W(\lambda_1\,|\,\lambda^1)\,d\lambda_1 = \delta(\lambda-\lambda^1) \tag{42}$$

The solution in terms of Fourier transforms is elementary. Introduce

$$\begin{aligned} \tilde{W}(x\,|\,\lambda^1) &= \int e^{i\lambda x}W(\lambda|\lambda^1)\,d\lambda^1 \\ W(\lambda\,|\,\lambda^1) &= \frac{1}{2\pi}\int e^{-i\lambda x}\tilde{W}(x\,|\,\lambda^1)\,dx \end{aligned} \tag{43}$$

Since $\tilde{G}_0(x\,|\,E) = \tilde{G}_0(-x\,|\,E)$, we have

$$\tilde{W}(x\,|\,\lambda^1) = \frac{e^{i\lambda^1x} + (\tau/2\pi)\tilde{G}_0(x\,|\,E)e^{i(2-\lambda^1)x}}{1 - \left[(\tau/2\pi)\tilde{G}_0(x\,|\,E)\right]^2} \tag{44}$$

Using the adjoint form

$$Y(\lambda)=\left[1-n\tau G_0(2-\lambda)\right]^{-1}\int W(\lambda\,|\,\lambda^1)G_0(\lambda^1-3)\,d\lambda^1 \tag{45}$$

we find

$$\Delta\bar{K}_1(2;3)=\frac{n\tau^3}{2\pi}\int\tilde{G}_0^2(x\,|\,E-n\tau)\tilde{G}_0(x\,|\,E)\frac{\left[e^{1k_2x}+(\tau/2\pi)\tilde{G}_0(x\,|\,E)\right]}{1-\left[(\tau/2\pi)\tilde{G}_0(x\,|\,E)\right]^2}\,dx \tag{46}$$

This reduces to the previous expression as $\tau\to 0$. Of course, the exact solution involves the determination of a kernel $S(\lambda\,|\,\lambda^1)$ in place of $W(\lambda\,|\,\lambda^1)$. It obeys

$$\{1-n\tau G_0(2-\lambda)\}S(\lambda\,|\,\lambda^1)-\frac{\tau}{2\pi}\int G_0(2-\lambda-\lambda_1)S(\lambda_1\,|\,\lambda^1)\,d\lambda_1=\delta(\lambda-\lambda^1) \tag{47}$$

The preceding considerations do not handle the bound-state regions for moderate densities. It is likely that the preceding equation has an exact solution by analytic function techniques, but we have not found it. The problem may be approached analytically via stationary variational principles, e.g., the Schwinger variation principle.

Along with $Y(\lambda)$ we consider the auxiliary function $I(\lambda)$, which obeys

$$\left[1-n\tau G_0(2-\lambda)\right]I(\lambda)-\frac{\tau}{2\pi}\int G_0(2-\lambda-\lambda_1)I(\lambda_1)\,d\lambda_1=G_0(2-\lambda) \tag{48}$$

This is the same as the equation for $Y$, except for the inhomogeneous term. We construct the functional

$$J=\int G_0(2-\lambda)\,Y(\lambda)\,d\lambda\cdot\int d\lambda_1\,G_0(3-\lambda_1)I(\lambda_1)/M \tag{49}$$

$$M=\int d\lambda\,I(\lambda)\left[1-n\tau G_0(2-\lambda)\right]Y(\lambda)-\frac{\tau}{2\pi}\int\int I(\lambda)G_0(2-\lambda-\lambda_1)\,Y(\lambda_1)\,d\lambda\,d\lambda_1 \tag{50}$$

$J$ is stationary for approximations to $I$ and $Y$ in the vicinity of the exact values. The stationary value of $J$ is $\int G_0(2-\lambda)\,Y(\lambda)\,d\lambda$, i.e., just the quantity needed to compute $\bar{K}_1(2;3)$. The simplest trial functions are obtained by inserting

$$Y(\lambda)=G_0(3-\lambda)/\left[1-n\tau G_0(2-\lambda)\right],\qquad I(\lambda)=G_0^*(2-\lambda) \tag{51}$$

to improve the weak scattering limit. Appropriate expressions from the Fourier transform solution may be used to extend the low-density case.

The results of the present approach are most naturally compared with Klauder's early, pioneering paper. Klauder has compared the density of states corresponding to several levels of approximation with the exact results of Frisch and Lloyd. It is relatively easy to do this with self-energies that are independent of $k_2$. Klauder's most sophisticated approximation takes into account strong $t$-matrix scattering, as well as medium effects, and leads to a momentum-dependent self-energy. In view of the complication, this approximation was not evaluated numerically and compared with the exact results.

In general it is easier to compute with the present type of theory since we work with explicit expressions for the self-energy. There is no difficulty connected with the computation of self-consistent propagators. As noted earlier, the restricted 2BA is close to Klauder's fifth approximation with $\Sigma$ approximated by $nt$ at an appropriate point. The more interesting general 2BA yields a $\Sigma$ that is accurate to $t^4$ and is therefore momentum dependent. We have not carried out any numerical computations with it. Our confidence in this expression is based on the agreement to order $t^4$ and on the general logical structure of the theory.

Thus far we have neglected the collective terms. It is easy to see that for the uncorrelated case they do not disturb the $t^4$ accuracy of the general 2BA. The hierarchy equation for $\bar{U}_2(\lambda)$, viz., Eq. (29), is exact for the uncorrelated system. The collective contribution to the source term is in general of order $t^2$ but vanishes for the uncorrelated system. An improved microscopic assumption for $\Gamma_\alpha$ involves a collective term that starts as $t^2$ and so could only give contributions to $\bar{U}_2$ from $\bar{B}_2$ and $\bar{C}_2$ that are of order $t^3$. Thus there are no collective contributions to the $t^4$ self-energy. In fact, the argument goes further. The improved $\Gamma_\alpha$ that results from adding a collective term to the general 2BA (as was done for the restricted 2BA in I), gives zero contributions to $\bar{U}_2$ from the $\bar{B}_2$ and $\bar{C}_2$ terms in the $N \to \infty$ limit.

The type of theory described here is expected to share one shortcoming of standard multiple scattering theories. It is not expected to describe the effects of very large fluctuations that result in deep traps. These have been treated by Lifshitz,[3] Halperin and Lax,[4] Zittarz and Langer,[5] and subsequently by many others.[6] They account for the tail in the density of states. They are of course contained in the exact Frisch–Lloyd[7] one-dimensional solutions.

## 4. UNCORRELATED SYSTEM—GENERAL SCATTERING MATRIX

We now briefly study the general 2BA for a general $\langle 1|t|3\rangle$ and for a separable three-dimensional $t$ matrix. For the uncorrelated system the truncation again implies that $\bar{U}_3 = 0$ and that $\bar{C}_2 = 0$. The equations for $\bar{U}_2(\lambda)$ are Eqs. (27) and (29).

The equation for $\lambda = 2-3$ again decouples in the $N \to \infty$ limit, and we have the same result as for the restricted 2BA:

$$\langle 1|\bar{U}_2(2-3)|3\rangle = N^2\langle 1|tG_0^*|3\rangle \tag{52}$$

For $\lambda \neq 2-3$ we solve $\langle 2-\lambda|\bar{U}_2(\lambda)|3\rangle$ and eliminate it. This gives

$$\begin{aligned}\langle 1|\bar{U}_2(\lambda)|3\rangle &- \langle 1|t_0(2-\lambda)|2-\lambda-\underline{\lambda}_1\rangle \\ &\times G_0(2-\lambda-\underline{\lambda}_1)\langle 2-\lambda-\underline{\lambda}_1|\bar{U}_2(\underline{\lambda}_1)|3\rangle \\ &= \langle 1|t_0(2-\lambda)|3-\lambda\rangle G_0(3-\lambda)N^2\langle 3-\lambda|tG_0^*|3\rangle \end{aligned} \tag{53}$$

where

$$\langle 1|t_0(2-\lambda)|4\rangle = \langle 1|t|4\rangle + N\langle 1|tG_0^*|2-\lambda\rangle\langle 2-\lambda|t|4\rangle \tag{54}$$

We are interested in computing

$$\begin{aligned}\langle 1|\bar{K}_1(2)|3\rangle &= \frac{\langle 1|tG_0|3\rangle}{N}\langle 3|\bar{U}_2(2-3)|3\rangle \\ &+ \frac{\langle 1|tG_0|2-\underline{\lambda}\rangle}{N}\langle 2-\underline{\lambda}|\bar{U}_2(\underline{\lambda})|3\rangle \end{aligned} \tag{55}$$

In the weak scattering limit we have

$$\begin{aligned}\langle 1|\bar{K}_1(2)|3\rangle &= N\langle 1|tG_0|3\rangle\langle 3|tG_0^*|3\rangle \\ &+ N\langle 1|tG_0^*|2-\underline{\lambda}\rangle\langle 2-\underline{\lambda}|tG_0|3-\underline{\lambda}\rangle\langle 3-\underline{\lambda}|tG_0^*|3\rangle \end{aligned} \tag{56}$$

This provides $t^4$ accuracy for the self-energy.

The integral equation for $\langle 1|\bar{U}_2(\lambda)|3\rangle$ is formidable for the general nonseparable $\langle 1|t|2\rangle$. It is, however, easy to reduce it to simple terms for a three-dimensional separable scattering potential

Let

$$\langle 1|t|3\rangle = \gamma u(1)u(3) \tag{57}$$

$$\int u^2(\lambda)\,d\lambda = 1$$

$u(\lambda)$ is a form factor and $\gamma$ measures the strength of the potential. We can simplify the integral equation with the ansatz

$$\langle 1|\bar{U}_2(\lambda)|3\rangle = \gamma^2 N^2 G_0^*(3)u(1)u(3)D(\lambda) \tag{58}$$

Then

$$D(\lambda)\{1 - N\gamma\mathcal{G}_0(2-\lambda)\} - \gamma\mathcal{G}_0(2-\lambda-\underline{\lambda})D(\underline{\lambda}_1) = \mathcal{G}_0(3-\lambda) \tag{59}$$

where

$$\mathcal{G}_0(\lambda) = u^2(\lambda)G_0(\lambda) \tag{60}$$

This is the same form as was encountered for $Y(\lambda)$ in the one-dimensional delta-function case. We have

$$\langle 1|\bar{U}_2(2-3)|3\rangle = N^2\gamma u(1)u(3)G_0^*(3) \tag{61}$$

$$\langle 1|\bar{K}_1(2)|3\rangle = N\gamma^2 G_0^*(3)u(1)u(3)\{\mathcal{G}_0(3) + \gamma u^2(2-\underline{\lambda})G_0(2-\underline{\lambda})D(\underline{\lambda})\} \tag{62}$$

We can find explicit expressions for the weak scattering and low-density limits in the same way as in the previous section.

## 5. HIGHER APPROXIMATIONS

For the uncorrelated case, we have seen that the general 2BA is equivalent to setting $\tilde{U}_3(-\lambda \| \lambda_1)$ equal to zero. It is instructive to study improvements by examining the second hierarchy equation. For simplicity we work with the delta-function potential and ignore collective effects. They will not contribute to the hierarchy equations.

For the two-point function we found that $\overline{U}_2(2-3)$ plays a special role since it has a direct source term. The $\overline{U}_2(\lambda)$ for $\lambda \neq 2-3$ is $tG_0(3)\overline{U}_2(2-3)$. We first find a more accurate expression for $\overline{U}_2(2-3)$. The exact first hierarchy equation is

$$\overline{U}_2(2-3)\left[1-NtG_0(3)\right] = N^2tG_0(3) + tG_0(2-\underline{\lambda}_1)$$
$$\overline{U}_3(3-2 \| \underline{\lambda}_1)\Delta(\underline{\lambda}_1 | \lambda)\Delta(\underline{\lambda}_1 | 2-3) \quad (63)$$

We therefore set down an equation for $\overline{U}_3(3-2 \| \lambda_1)$. In the $N \to \infty$ limit

$$\left\{1-NtG_0(2-\lambda_1)\right\}\overline{U}_3(3-2 \| \lambda_1) - tG_0(\underline{\lambda}_2+\lambda_1+3-2-2)\overline{U}_3(3-2 \| \underline{\lambda}_2)$$
$$= NtG_0(2-\lambda_1)\overline{U}_2(2-3) + tG_0(2-\underline{\lambda}_2)\Delta(\underline{\lambda}_2 | \lambda_1)$$
$$\times \overline{V}_4(3-2|2-3-\lambda_1|\lambda_1+\underline{\lambda}_2 \| \underline{\lambda}_2) \quad (64)$$

where

$$V_4 = \Sigma' E_\gamma(3-2)E_\beta(2-3-\lambda_1)E_\alpha(\lambda_1-\lambda_2)E_\delta(\lambda_2)\Gamma_\delta \quad (65)$$

with all of the particle indices distinct.

In the approximation that we neglect $\overline{V}_4$, this is a self-contained equation. It may be solved with a kernel:

$$\overline{U}_3(3-2 \| \lambda_1) = Nt\mathbb{S}(\lambda_1 | \underline{\lambda}_2)G_0(2-\underline{\lambda}_2)\overline{U}_2(2-3) \quad (66)$$

where $\mathbb{S}(\lambda_1 | \lambda_2)$ obeys.

$$\left[1-NtG_0(2-\lambda_1)\right]\mathbb{S}(\lambda_1 | \lambda_2) - tG_0(\underline{\lambda}_3+\lambda_1+3-2-2)\mathbb{S}(\underline{\lambda}_3 | \lambda_2)$$
$$= \delta(\lambda_1 | \lambda_2) \quad (67)$$

This is essentially the same as the integral equation of Section 3. Thus we have

$$\overline{U}_2(2-3)\left[1-NtG_0(3)-Nt^2G_0(2-\underline{\lambda}_1)\mathbb{S}(\underline{\lambda}_1 | \underline{\lambda}_2)G_0(2-\underline{\lambda}_2)\right] = N^2tG_0(3)$$
$$(68)$$

The additional power of $t$ in this result gives a $t^4$ contribution to $\bar{K}_1(2;3)$ and thus a $t^5$ contribution to the self-energy.

Equation (29) is the first hierarchy equation for $\bar{U}_2(\lambda)$ when $\lambda \neq 2-3$. Here $\bar{U}_2(2-3)$ is a given homogeneous term. Thus we need an equation for $\bar{U}_3(-\lambda \| \lambda_1)$. After appropriate particle and wave vector reductions we find

$$\begin{aligned}\{1 - NtG_0(2-\lambda_1)\}\bar{U}_3(-\lambda \| \lambda_1) - t\Big[G_0(2-\lambda_1+\lambda-\underline{\lambda}_2)\bar{U}_3(-\lambda \| \underline{\lambda}_2) \\ + G_0(2-\lambda-\underline{\lambda}_2)\bar{U}_3(\lambda-\lambda_1 \| \underline{\lambda}_2)\Big] \\ = NtG_0(2-\lambda_1)\Big[\bar{U}_2(\lambda)+\bar{U}_2(\lambda_1-\lambda)\Big] \\ + tG_0(3-\lambda_1)\Big[\bar{U}_3(3-2 \| \lambda+2-3)+\bar{U}_3(3-2 \| \lambda_1-\lambda+2-3)\Big] \\ + tG_0(2-\underline{\lambda}_2)\bar{V}_4(\lambda_1-\lambda | -\lambda | \lambda_1-\underline{\lambda}_2 \| \underline{\lambda}_2)\Delta(\underline{\lambda}_2 | \lambda_1)\end{aligned} \tag{69}$$

We have isolated the terms involving $\bar{U}_3(3-2 \| \lambda_1)$, since they are $N$ times $\bar{U}_3(-\lambda \| \lambda_1)$ for $\lambda \neq 2-3$. We can now make the truncation $\bar{V}_4 = 0$. This leads to a difficult but still tractable self-contained equation for the $\bar{U}_3(-\lambda \| \lambda_1)$. However, an even cruder approximation still extends the accuracy well beyond the general 2BA. In lowest order, we neglect the integral terms on the left-hand side of the above equation. Then

$$\begin{aligned}\bar{U}_3(-\lambda \| \lambda_1) \approx tG_0^*(2-\lambda_1)\Big[\bar{U}_2(\lambda)+\bar{U}_2(\lambda_1-\lambda)\Big] \\ + \frac{tG_0(3-\lambda_1)}{1-NtG_0(2-\lambda_1)}\Big[\bar{U}_3(3-2 \| \lambda+2-3) \\ + \bar{U}_3(3-2 \| \lambda_1-\lambda+2-3)\Big]\end{aligned} \tag{70}$$

This leads to a modified integral equation for $\bar{U}_2(\lambda)$.

$$\begin{aligned}\bar{U}_2(\lambda)\Big[1 - NtG_0(2-\lambda) - Nt^2G_0^*(\underline{5})G_0(\underline{5})\Big] \\ - tG_0^*(2-\lambda-\underline{\lambda}_1)\bar{U}_2(\underline{\lambda}_1)\Delta(\underline{\lambda}_1 | 2-3) \\ = tG_0(3-\lambda)\bar{U}_2(2-3) + Nt^2G_0^*(2-\underline{\lambda}_1) \\ \times G_0(3-\underline{\lambda}_1)\Big[\mathbb{S}(\lambda+2-3 | \underline{\lambda}_2) + \mathbb{S}(\lambda_1-\lambda+2-3 | \underline{\lambda}_2)\Big] \\ \times G_0(2-\underline{\lambda}_2)\bar{U}_2(2-3)\end{aligned} \tag{71}$$

This modified equation for $\bar{U}_2(\lambda)$ is no more difficult to solve than the general 2BA equation. The only extra complication is the extra homogeneous term.

We now estimate the error in the self-energy that is involved in using Eq. (66) for $\bar{U}_2(2-3)$ and Eq. (71) for $\bar{U}_2(\lambda)$. The error in $\bar{U}_3(-\lambda \| \lambda_1)$ arising from using Eq. (70), i.e., neglecting the integral term, is of order $t^4$.

This contributes an error of order of $t^5$ in $\overline{U}_2(\lambda)$ and so an error $t^7$ in the self-energy. The error in $\overline{U}_2(2-3)$ may be obtained by noting that $\overline{V}_4$ is of order $t\overline{U}_3(3-2\,\|\,\lambda_1) \sim t^2\overline{U}_2(2-3) \sim t^3$. Thus $\overline{U}_3(3-2\,\|\,\lambda_1)$ is in error of order $t^4$, $\overline{U}_2(2-3)$ in error of order $t^5$, $\overline{K}_1$ of order $t^6$ and the self-energy in error of order $t^7$.

Thus far our result for the self-energy is accurate through terms of order $t^6$. We still have to verify that the collective contributions to the source term $J_\alpha$ and to the kernel $L_{\alpha\beta}$ do not disturb this. We start by examining the errors in the three-point functions. We note that the collective part of the source term vanishes for both $\overline{U}_3(3-2\,\|\,\lambda_1)$ and $\overline{U}_3(-\lambda\,\|\,\lambda_1)$. The contribution of the collective part of the kernel to $\overline{U}_3(3-2\,\|\,\lambda_1)$ is

$$\frac{1}{N}\left\langle 1\left|\frac{\overline{K}_0}{1-\overline{K}_0}\,tG_0\right|2-\underline{\lambda}_2\right\rangle$$
$$\times\langle 2-\underline{\lambda}_2|\,\overline{E_3(2-3-\lambda_1\,|\,3-2\,|\,\lambda_1)\sum_\alpha E^1_\alpha(\lambda_2)L_{\alpha\alpha_1}\Gamma_{\alpha_1}}\,|3\rangle \quad (72)$$

since the source term vanishes. This is of order $t^4$ since $\overline{K}_0$, $L_{\alpha\beta}$, and $\Gamma_\alpha$, are all of $\sim t$. The same thing is true for $\overline{U}_3(-\lambda\,\|\,\lambda_1)$. In summary, the collective terms do not contribute to the self-energy through order $t^6$. For uncorrelated one-dimensional delta-function potentials the self-energy to order $t^6$ is given by

$$\Sigma(2) = n\tau\left[1-\overline{K}_1(2;3)\right]^{-1}$$
$$\overline{K}_1(2;3) = \frac{1}{N}\,tG_0(3)\,\overline{U}_2(2-3) + \frac{t}{N}\,G_0(2-\underline{\lambda})\Delta(\underline{\lambda}\,|\,2-3)\,\overline{U}_2(\underline{\lambda}) \quad (73)$$

with $\overline{U}_2(2-3)$ given by Eq. (67) and (68). $\overline{U}_2(\lambda)$ is given by Eq. (71).

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